CHAPTER 11:

CYLINDRICAL COORDINATES

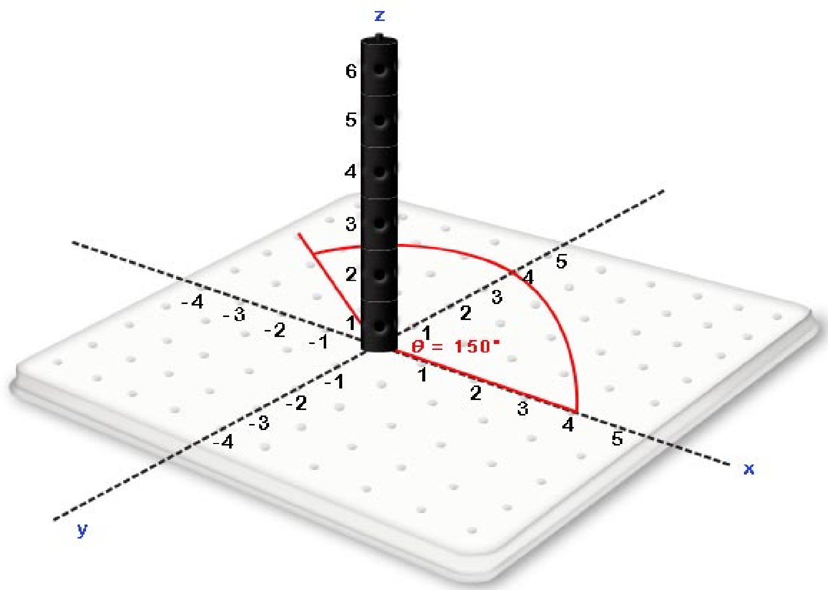
11.1 DEFINITION OF CYLINDRICAL COORDINATES

A location in 3-space can be defined with (r, θ, z) where

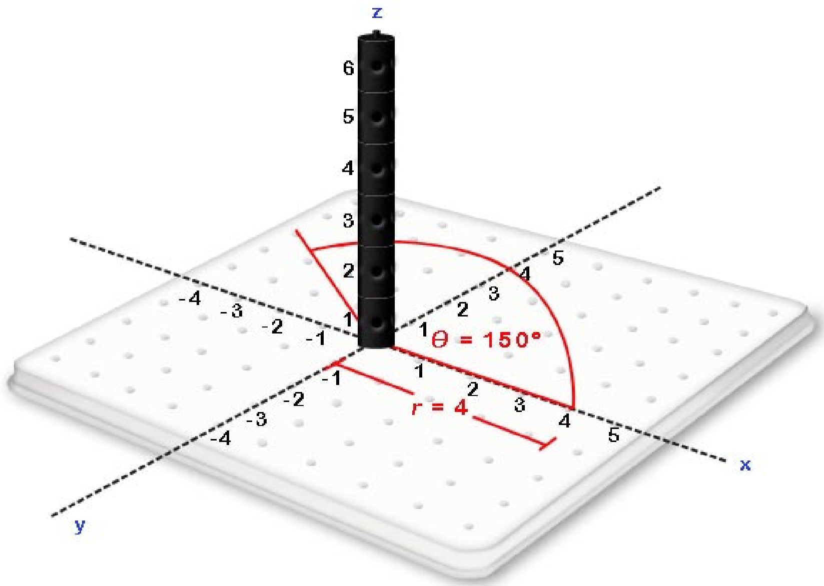
- (r, θ) is a location in the xy plane defined in polar coordinates and
- z is the height in units over the location (r, θ)in the xy plane

Example Exercise 11.1.1: Find the point $(r, \theta, z) = (150°, 4, 5)$.

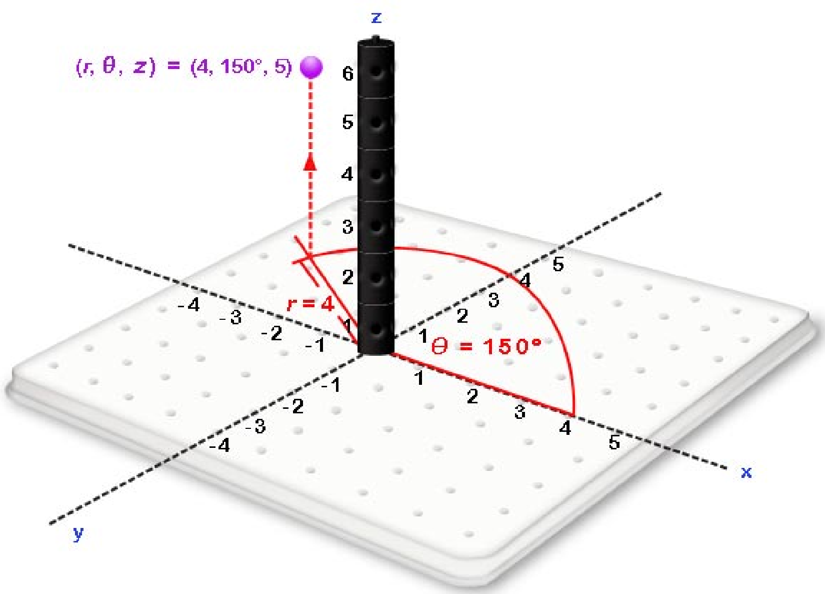
Solution: Starting with $\theta = 150°$ we can obtain all the points consistent with $\theta = 150°$ on the xy plane.



Of those points where $\theta = 150°$ on the xy plane, we can find the point that is 4 units away from the origin. I.e., the point consistent with $r = 4$.

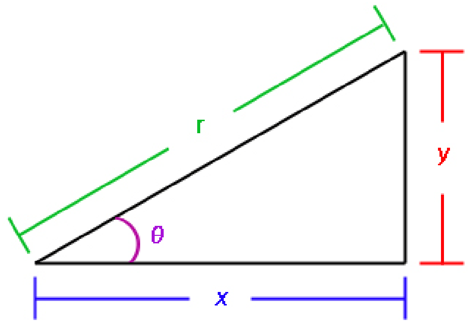


If we move 5 units directly upward from (r, θ) = (4, 150°) in the xy plane, we obtain the point (r, θ, z) = (4, 150°, 5).



11.2 CONVERTING BETWEEN CUBIC AND CYLINDRICAL COORDINATES

As both (x, y) and (r, θ) reside in the xy plane we can use the same diagram we used in chapter 9 that shows a location in both rectangular and polar coordinates.



From this diagram, the following relations obtained with polar coordinates remain unchanged:

- $x = rcos(\theta)$,
- $y = rsin(\theta)$,
- $x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$ and
- $\tan(\theta) = \frac{x}{y}$ or $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ when (x, y) is in the first or forth quadrant and $\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$ when the point (x, y) is in the second or third quadrant

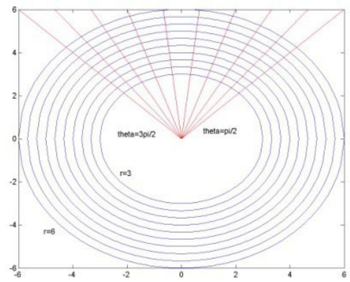
The variable z is the height over the xy plane in both coordinate systems and correspondingly remains unchanged.

11.3 CYLINDRICAL CUBES

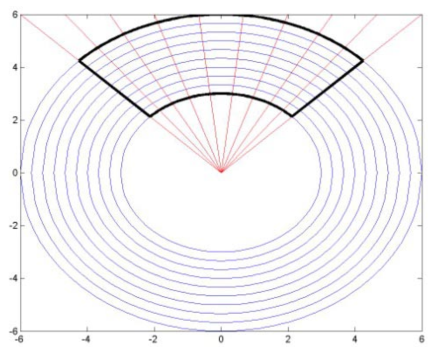
We refer to (x, y, z) as cubic coordinates since the set of points satisfying $a \leq x \leq b$, $c \leq y \leq d$ and $e \leq z \leq f$ where a, b, c, d, e and f are all constants will form a cube. Hence, our expectation is with appropriate selection of the constants a, b, c, d, e and f, the set of points satisfying $a \leq r \leq b$, $c \leq \theta \leq d$ and $e \leq z \leq f$ will represent either a cylinder or a cylindrical solid.

Example Exercise 11.3.1: Find the cylindrical solid associated with $3 \leq r \leq 6, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ and $2 \leq z \leq 5$.

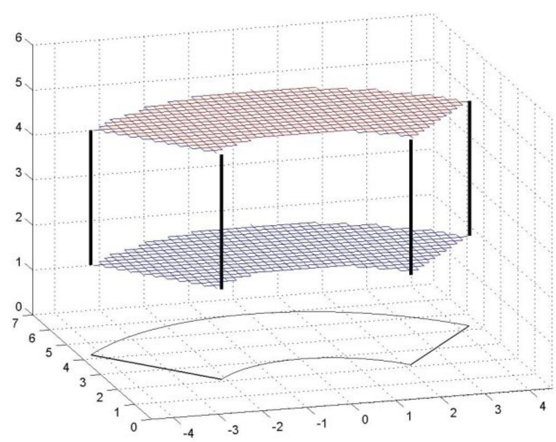
Solution:



In the above graph, the set of points where r = 3, r = 6 and r equal to various values between 3 and 6 are shown in blue. The set of points where $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$ and θ equal to various values between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are shown in red. The set of points where both $3 \leq r \leq 6$ and $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ will be the intersection of the blue and red regions as is outlined in black in the following diagram.



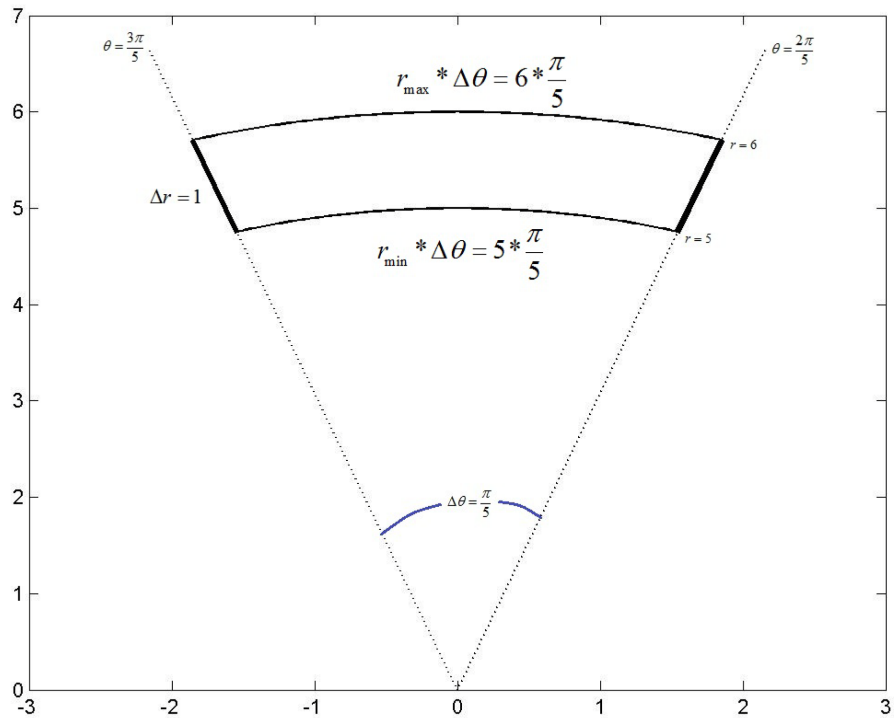
For each point in the polar region defined in the *xy* plane above, we wish all points with a height between $z = 2$ and $z = 5$. This will result in the cylindrical cube shown below.



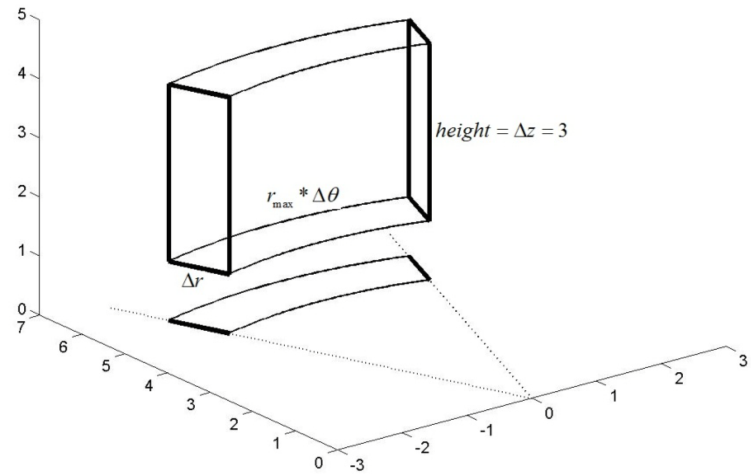
11.4 APPROXIMATING THE VOLUME OF A CYLINDRICAL CUBE

The region associated with the set of points where *r*, θ, and *z* go from one constant to another constant is often referred to as a cylindrical cube

If we consider the cylindrical cube $5 \le r \le 6$, $\frac{2\pi}{5} \le \theta \le \frac{3\pi}{5}$ and $1 \le z \le 4$ we can start with the polar rectangle $5 \le r \le 6$, $\frac{2\pi}{5} \le \theta \le \frac{3\pi}{5}$ shown in the following diagram. Remember from section 9.5 that as $\Delta r \to 0$ and $\Delta\theta \to 0$, a polar rectangle will come to resemble a rectangle with area length times width. As a result, it is not uncommon to approximate the area of a polar rectangle with length times width.



In the polar rectangle above, the top and bottom sides are two arcs with angle $\Delta\theta = \frac{\pi}{5}$. The bottom arc has radius $r = 5$ and the top arc has radius $r = 6$. Remembering that the length of an arc is given by the angle of the arc measured in radians multiplied by the radius of the arc, we find that the bottom arc has length $5 * \frac{\pi}{5} = \pi$ and the top arc has length $6 * \frac{\pi}{5}$. Both the left and right sides both have $\Delta r = 1$. Hence if we use smallest *r* associated with the region, the approximate area of the polar rectangle will be $length * width = \pi * 1$. If we use the largest *r* associated with the region, the approximate area of the polar rectangle will be $ngth * width = \left(6 * \frac{\pi}{5}\right) * 1$. The set of points (*x*, *y*) in this polar rectangle with values of *z* satisfying $1 \le z \le 4$ will produce the cylindrical cube shown in the following diagram. As the area of the base is approximated by $length * width$, the volume of this cube can be approximated by $length * width * height$. Using $r = 5$, this will produce $Volume = \pi * 1 * 3$. Using $r = 6$, this will produce $Volume = 6 * \frac{\pi}{5} * 1 * 3$.



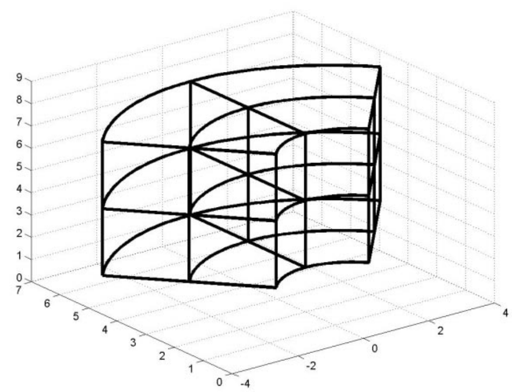
In general, we can approximate the volume of the cylindrical cube $r_1 \leq r \leq r_2$, $\theta_1 \leq \theta \leq \theta_2$ and $z_1 \leq z \leq z_2$ with $Volume = R\Delta r\Delta\theta\Delta z$ with $R = r_1$, $R = r_2$ or R being any r associated with the region that we have selected. As $\Delta r \longrightarrow 0$ and $\Delta\theta \longrightarrow 0$, this approximation becomes precise.

11.5 USING RIEMANN SUMS AND THE FUNDAMENTAL THEOREM TO OBTAIN THE MASS OF CYLINDRICAL CUBES

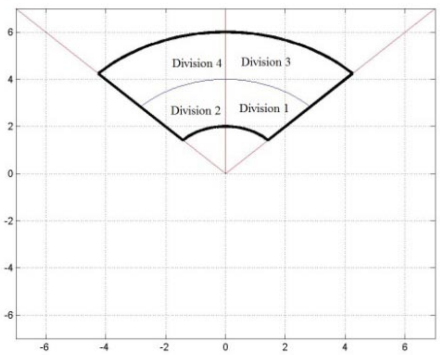
Example Exercise 11.5.1: A cylindrical cube $2 \leq r \leq 6$, $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ and $2 \leq z \leq 8$ has *density* = $(r\cos(\theta) + z)\frac{kg}{m^3}$. Use Riemann Sums with two divisions in r, θ and z to approximate the mass of the cube using the smallest value of each variable to represent a given division. Then express the Riemann Sum as a triple summation and use the fundamental theorem to find the precise mass of the cube.

Solution:

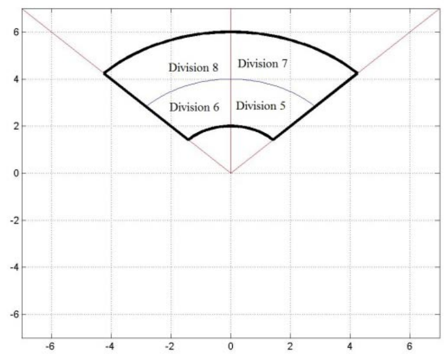
Step 1: Divide r, θ and z into two parts r_1, r_2, θ_1, θ_2, z_1, z_2, Δr, $\Delta\theta$ and Δz:



The divisions in the preceding diagram can also be displayed as two stories which will allow us to view the divisions in the *xy* plane.



First Story: $2 \le z \le 5$

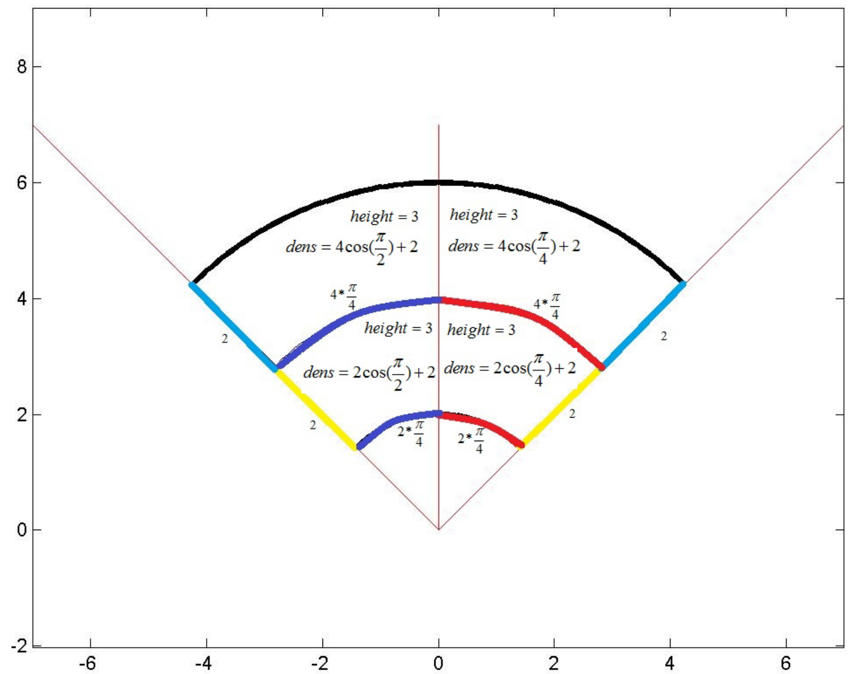


Second Story: $5 \le z \le 8$

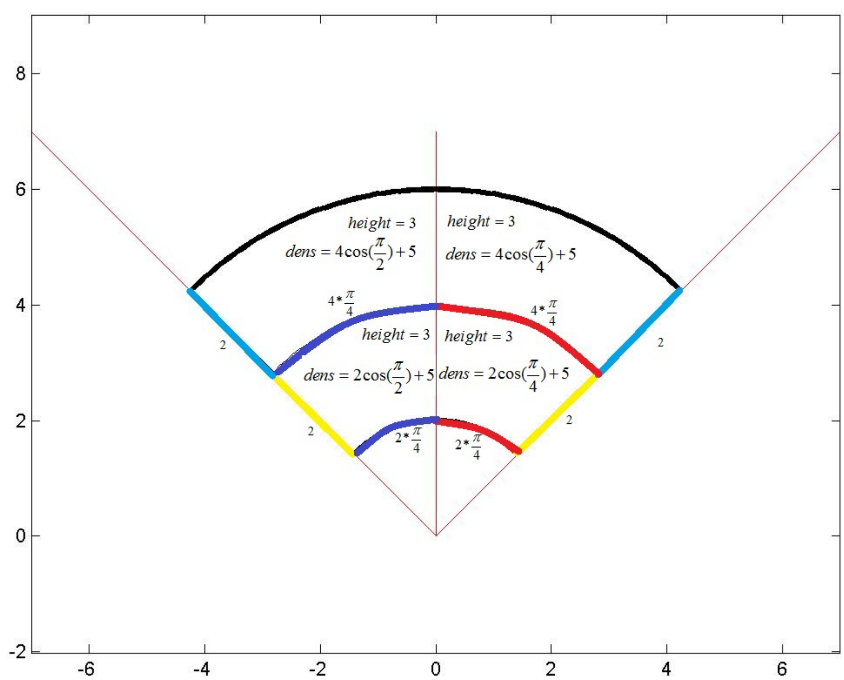
In divisions 1, 2, 5 and 6, r goes from 2 to 4 and in divisions 3, 4, 7 and 8; r goes from 4 to 6. In divisions 1, 3, 5 and 7, θ goes from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ and in divisions 2, 4, 6 and 8, θ goes from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$. In divisions 1, 2, 3 and 4, z goes from 2 to 5 and in divisions 5, 6, 7 and 8; z goes from 5 to 8. Hence, $r_1 = 2$, $r_2 = 4$, $\theta_1 = \frac{\pi}{4}$, $\theta_2 = \frac{\pi}{2}$, $z_1 = 2$, $z_2 = 5$, $\Delta r = 2$, $\Delta\theta = \frac{\pi}{4}$ and $\Delta z = 3$.

Step 2: Find the appropriate numeric approximation for the length, width, height, volume, density and mass for each division:

In Section 11.4, we approximated the volume of a polar cube with *volume = length*width*height* where width = Δr, *length* = $r\Delta\theta$ and *height* = Δz. The density of each division is given by *density* =($r\cos(\theta) + z$). Using the minimum value of each variable in each division, the following two diagrams show the length, width, height and density for the 4 divisions on the first floor and the four divisions on the second floor.



First Floor: $2 \le z \le 5$



Second Floor: $5 \le z \le 8$

The length, width, height, volume, density and mass of each division are summarized in the following table:

Division	Length (m)	Width (m)	Height (m)	Volume (m^3)	Density $\left(\frac{kg}{m^3}\right)$	Mass (kg)
1	$2*\frac{\pi}{4}=\frac{\pi}{2}$	2	3	3π	$2cos\left(\frac{\pi}{4}\right)+2$	$(2cos\left(\frac{\pi}{4}\right)+2)\,3\pi$
2	$2*\frac{\pi}{4}=\frac{\pi}{2}$	2	3	3π	$2cos\left(\frac{\pi}{2}\right)+2$	$(2cos\left(\frac{\pi}{4}\right)+2)\,3\pi$
3	$4*\frac{\pi}{4}=\pi$	2	3	6π	$4cos\left(\frac{\pi}{4}\right)+2$	$(4cos\left(\frac{\pi}{2}\right)+2)\,6\pi$
4	$4*\frac{\pi}{4}=\pi$	2	3	6π	$4cos\left(\frac{\pi}{4}\right)+2$	$(4cos\left(\frac{\pi}{2}\right)+2)\,6\pi$
5	$2*\frac{\pi}{4}=\frac{\pi}{2}$	2	3	3π	$2cos\left(\frac{\pi}{4}\right)+5$	$(2cos\left(\frac{\pi}{4}\right)+5)\,3\pi$
6	$2*\frac{\pi}{4}=\frac{\pi}{2}$	2	3	3π	$2cos\left(\frac{\pi}{2}\right)+5$	$(2cos\left(\frac{\pi}{4}\right)+5)\,3\pi$
7	$4*\frac{\pi}{4}=\pi$	2	3	6π	$4cos\left(\frac{\pi}{4}\right)+5$	$(4cos\left(\frac{\pi}{2}\right)+5)\,6\pi$
8	$4*\frac{\pi}{4}=\pi$	2	3	6π	$4cos\left(\frac{\pi}{2}\right)+5$	$(4cos\left(\frac{\pi}{2}\right)+5)\,6\pi$

Step 3: Add the masses of the eight divisions to approximate the total mass of the solid:

$$\boldsymbol{Mass \approx \left(2cos\left(\frac{\pi}{4}\right)+2\right)3\pi+\left(2cos\left(\frac{\pi}{2}\right)+2\right)3\pi+\left(4cos\left(\frac{\pi}{4}\right)+2\right)6\pi+\left(4cos\left(\frac{\pi}{2}\right)+2\right)6\pi+\left(2cos\left(\frac{\pi}{4}\right)+5\right)3\pi+\left(2cos\left(\frac{\pi}{2}\right)+5\right)3\pi+\left(4cos\left(\frac{\pi}{4}\right)+5\right)6\pi+\left(4cos\left(\frac{\pi}{2}\right)+5\right)6\pi}$$

Step 4: Repeat Step 2 using r_1, r_2, θ_1, θ_2, z_1, z_2, Δr, $\Delta\theta$ and Δz instead of numerical values as appropriate. The volume column has been left out of the following table for ease of viewing.

Division	Length (m)	Width (m)	Height (m)	Density $\left(\frac{kg}{m^3}\right)$	Mass (kg)
1	$r_1\Delta\theta$	Δr	Δz	$r_1cos(\theta_1)+z_1$	$(r_1cos(\theta_1)+z_1)\,r_1\,\Delta\theta\Delta r\Delta z$
2	$r_1\Delta\theta$	Δr	Δz	$r_1cos(\theta_2)+z_1$	$(r_1cos(\theta_2)+z_1)\,r_1\,\Delta\theta\Delta r\Delta z$
3	$r_2\Delta\theta$	Δr	Δz	$r_2cos(\theta_1)+z_1$	$(r_2cos(\theta_1)+z_1)\,r_2\,\Delta\theta\Delta r\Delta z$
4	$r_2\Delta\theta$	Δr	Δz	$r_2cos(\theta_2)+z_1$	$(r_2cos(\theta_2)+z_1)\,r_2\,\Delta\theta\Delta r\Delta z$
5	$r_1\Delta\theta$	Δr	Δz	$r_1cos(\theta_1)+z_2$	$(r_1cos(\theta_1)+z_2)\,r_1\,\Delta\theta\Delta r\Delta z$
6	$r_1\Delta\theta$	Δr	Δz	$r_1cos(\theta_2)+z_2$	$(r_1cos(\theta_2)+z_2)\,r_1\,\Delta\theta\Delta r\Delta z$
7	$r_2\Delta\theta$	Δr	Δz	$r_2cos(\theta_1)+z_2$	$(r_2cos(\theta_1)+z_2)\,r_2\,\Delta\theta\Delta r\Delta z$
8	$r_2\Delta\theta$	Δr	Δz	$r_2cos(\theta_2)+z_2$	$(r_2cos(\theta_2)+z_2)\,r_2\,\Delta\theta\Delta r\Delta z$

Step 5: Add the masses of the eight divisions to approximate the total mass of the solid parts r_1, r_2, θ_1, θ_2, z_1, z_2, Δr, $\Delta\theta$ and Δz:

$Mass \approx (r_1 \cos(\theta_1) + z_1)\, r_1\, \Delta\theta\Delta r\Delta z + (r_1 \cos(\theta_2) + z_1)\, r_1\, \Delta\theta\Delta r\Delta z + (r_2 \cos(\theta_1) + z_1)\, r_2\, \Delta\theta\Delta r\Delta z + (r_2 \cos(\theta_2) + z_1)\, r_2\, \Delta\theta\Delta r\Delta z + (r_1 \cos(\theta_1) + z_2)\, r_1\, \Delta\theta\Delta r\Delta z + (r_1 \cos(\theta_2) + z_2)\, r_1\, \Delta\theta\Delta r\Delta z + (r_2 \cos(\theta_1) + z_2)\, r_2\, \Delta\theta\Delta r\Delta z + (r_2 \cos(\theta_2) + z_2)\, r_2\, \Delta\theta\Delta r\Delta z$

Step 6: Express the approximate mass in Step 5 as a triple summation:

$Mass \approx [(r_1 \cos(\theta_1) + z_1)\, r_1\, \Delta\theta\Delta r\Delta z + (r_1 \cos(\theta_2) + z_1)\, r_1\, \Delta\theta\Delta r\Delta z] + [(r_2 \cos(\theta_1) + z$
$r\Delta z]$

By grouping the [..], we obtain:

$$Mass \approx \left\{\sum_{k=1}^{2}(r_1\cos(\theta_k) + z_1)r_1\Delta\theta\Delta r\Delta z + \sum_{k=1}^{2}[(r_2\cos(\theta_k) + z_1)r_2\Delta\theta\Delta r\Delta z\}] + \left\{\sum_{k=1}^{2}(r_1\cos(\theta_k) + z_2)\right.\right.$$

By grouping the {..}, we obtain:

$Mass \approx \sum_{j=1}^{2}\sum_{k=1}^{2}(r_j cos(\theta_k) + z_1)r_j\, \Delta\theta\Delta r\Delta z + \sum_{j=1}^{2}\sum_{k=1}^{2}(r_j \cos(\theta_k) + z_2)r_j\, \Delta\theta\Delta r\Delta z$

Grouping these last two terms we obtain:

$Mass \approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(r_j\cos(\theta_k) + z_i)r_j\, \Delta\theta\Delta r\Delta z$

Step 7: As $\Delta r \to 0$, $\Delta\theta \to 0$ and $\Delta z \to 0$, this approximation becomes precise and we can apply the fundamental theorem to find the precise mass of the solid:

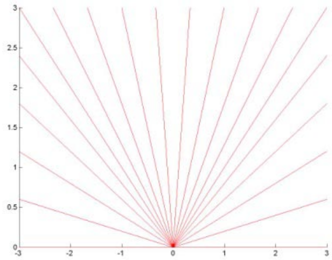
$Mass = \lim_{\Delta z\to 0} \lim_{\Delta r\to 0} \lim_{\Delta\theta\to 0} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(r_j\cos(\theta_k) + z_i)r_j\, \Delta\theta\Delta r\Delta z$

$Mass = \int_2^8 \int_2^6 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (r\cos(\theta) + z)r\; d\theta dr dz$

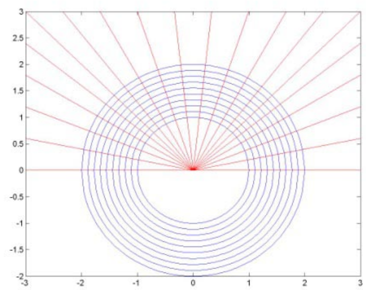
11.6 VOLUMES ASSOCIATED WITH INTEGRALS IN CYLINDRICAL COORDINATES

Example Exercise 11.6.1: Find the volume associated with $\int_0^{\pi}\int_1^2\int_{r^2}^4 r\; dz dr d\theta$

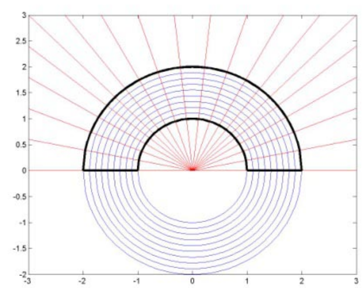
Solution: From section 9.7 we should remember that $\int\int r\; dr d\theta$ corresponds to $\sum\sum r_i\, \Delta\theta\Delta r$ which represents a region in the *xy* plane and hence the *r* inside of the integral is interpreted as part of the volume. Working from the outside inward, the first datum from the integral is $\int_0^{\pi} .. d\theta$ indicating that our volume will reside between $\theta = 0$ and $\theta = \pi$.



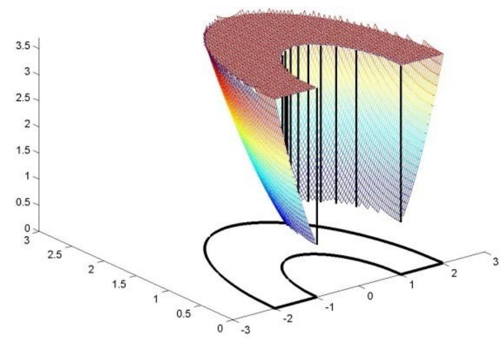
The second datum from the integral is $\int_1^2 dr$ indicating that for every value of θ between $\theta = 0$ and $\theta = \pi$, we will accept values of r that between $r = 1$ and $r = 2$.



The combination of the two data defined thus far give the following region in the *xy* plane.



The final datum from the integral is $\int_{r^2}^4 dz$. Hence, the volume represented by $\int_0^\pi \int_1^2 \int_{r^2}^4 r\, dz dr d\theta$ will be the volume above this region with floor equal to $z = r^2$ and ceiling equal to $z = 4$. Using the relations shown in 10.3, we can see that the surface above the region is $z = r^2 = x^2 + y^2$ which is a paraboloid that we have seen many times and that we should be familiar with. Hence, the solid associated with $\int_0^\pi \int_1^2 \int_{r^2}^4 r\, dz dr d\theta$ is shown in the following diagram.



EXERCISE PROBLEMS:

1) Express the following integrals in cylindrical coordinates.

A. $\int_{-2}^{2}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}\int_{x}^{2x+y+10} dzdydx$

B. $\int_{0}^{3}\int_{-\sqrt{9-x^2-y^2}}^{0}\int_{x^2+y^2}^{x^2+y^2+4} dzdydx$

C. $\int_{-4}^{0}\int_{0}^{\sqrt{16-x^2-y^2}}\int_{x^2+y^2}^{8-x^2-y^2}(x+y)dzdydx$

D. $\int_{0}^{2}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}\int_{x^2+y^2}^{8+x^2} x^2+y^2\ dzdydx$

E. $\int_{-6}^{0}\int_{-\sqrt{36-x^2-y^2}}^{\sqrt{36-x^2-y^2}}\int_{x^2+y^2}^{36} tan^{-1}\frac{y}{x}\ dzdydx$

F. $\int_{0}^{\sqrt{2}}\int_{x}^{\sqrt{4-x^2-y^2}}\int_{x^2+y^2}^{8-x^2-y^2}\left(\frac{y}{x}\right)\ dzdydx$

2) Express the following integrals in rectangular coordinates.

A. $\int_{0}^{\pi}\int_{0}^{2}\int_{1}^{10-r^2}(r^2)r\ drd\theta$

B. $\int_{\pi}^{2\pi}\int_{0}^{3}\int_{1+r^2}^{10} r^2cos(\theta)\ dzdrd\theta$

C. $\int_{0}^{2\pi}\int_{0}^{3}\int_{r^2}^{18-r^2}(r^3)^3\ dzdrd\theta$

D. $\int_{\frac{3\pi}{2}}^{2\pi}\int_{0}^{3}\int_{rcos(\theta)}^{3rcos(\theta)+5} r^4cos(\theta)\ dzdrd\theta$

E. $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{3}\int_{2rsin(\theta)+1}^{3rcos(\theta)+5} r^6sin(\theta)\ dzdrd\theta$

F. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{3}\int_{r}^{\sqrt{18-r^2}} r^6\ sin(\theta)\ dzdrd\theta$

3) Express the volume of the following solids as a triple integral in **(i)** cubic and **(ii)** cylindrical coordinates

A. The solid between $z = 1$ and $z = 10 - r^2$.
B. The solid between $z = 1 + r^2$ and $z = 10$.
C. The solid between $z = r^2$ and $z = 8 - r^2$.
D. The solid between $z = 3 + r^2$ and $z = 21 - r^2$.
E. The solid between $z = r$ and $z = \sqrt{18 + r^2}$.
F. The solid between $z = 4$ and $z = 13 - x^2 - y^2$.
G. The solid between $z = 2 + x^2 + y^2$ and $z = 27$.
H. The solid between $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$.
I. The solid between $z = 1 + x^2 + y^2$ and $z = 9 - x^2 - y^2$.
J. The solid between $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{32 - x^2 - y^2}$.

4) The density of a solid is $x + z \, \frac{\text{kg}}{\text{m}^3}$ and we wish to obtain the mass of the solid $1 \le r \le 7$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$ and $2 \le z \le 6$.

A. If there are two divisions in each variable and the number of fish is to be approximated using the minimum value for each variable in each division, find $r_1, r_2, \theta_1, \theta_2, z_1$ and z_2 use them to fill in the following table with numerical values.

Division	Length	Width	Height	Density	Mass
1					
2					
3					
4					
5					
6					
7					
8					

B. Use the values of $r_1, r_2, \theta_1, \theta_2, z_1$ and z_2 to fill in the same table below using r_1, r_2, θ_1, θ_2, z_1 and $z_2, \Delta r, \Delta\theta$ and Δz instead of numerical values. (Note, the divisions should not change between the two tables.)

Division	Length	Width	Height	Density	Mass
1					
2					
3					
4					
5					

6					
7					
8					

C. Express the approximate mass numerically.

D. Express the approximate mass using $r_1, r_2, \theta_1, \theta_2, z_1$ and z_2, Δr, $\Delta\theta$ and Δz.

E. Express the mass obtained in part D in the form $\sum\sum\sum (\ldots)\Delta z \Delta r \Delta\theta$.

F. Take the appropriate limits to convert the sum in part E to an integral and evaluate the integral to obtain the precise mass of the solid.

5) The density of fish in a cylindrical tank f is x^2+y^2+z $\frac{peces}{\mathrm{m}^3}$ and we wish to obtain the number of fish in the tank described by $2 \le r \le 4$, $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $0 \le z \le 4$.

A. If there are two divisions in each variable and the number of fish is to be approximated using the maximum value for each variable in each division, find $r_1, r_2, \theta_1, \theta_2, z_1$ and z_2 use them to fill in the following table with numerical values.

Division	Length	Width	Height	Density	No. of fish
1					
2					
3					
4					
5					
6					
7					
8					

B. Use the values of $r_1, r_2, \theta_1, \theta_2, z_1$ and z_2 to fill in the same table below using $r_1, r_2, \theta_1, \theta_2, z_1$ and z_2, Δr, $\Delta\theta$ and Δz instead of numerical values. (Note, the divisions should not change between the two tables.)

Division	Length	Width	Height	Density	No. of fish
1					
2					
3					
4					
5					
6					
7					
8					

C. Express the approximate no. of fish numerically.

D. Express the approximate number of fish using $r_1, r_2, \theta_1, \theta_2, z_1$ and z_2, Δr, $\Delta\theta$ and Δz

E. Express the number of fish obtained in part D in the form $\sum\sum\sum (\ldots)\Delta z \Delta r \Delta\theta$.

F. Take the appropriate limits to convert the sum in part E to an integral and evaluate the integral to obtain the precise number of fish.